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Newton's graphical method for central force orbits

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In the *Principia*, Book 1, Proposition 1, Newton gave a geometrical proof of angular momentum conservation for central forces. A geometrical construction associated with this proof provides a very efficient graphical method to obtain approximate orbital curves for such forces that can be traced with a ruler and a pencil. Newton's geometrical construction also satisfies a discrete form of energy conservation and it is time reversal invariant. An algorithm corresponding to this construction provides an efficient and stable numerical method to integrate the equations of motion of classical mechanics. In the past, this algorithm has been rediscovered several times without recognizing its connection to Newton's geometrical construction.
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I. INTRODUCTION

In Proposition 1 of the *Principia*, Book 1, Newton gave a geometrical proof of Kepler's area law for central forces.[1,2] This law, conjectured by Kepler from astronomical observations of planetary motion by Tycho Brahe, corresponds to the conservation of angular momentum for such forces.[3] The geometrical construction associated with Proposition 1 also provides a very efficient method to obtain, both graphically and numerically, approximate orbits for general central forces. Graphically, these orbits can be traced as polygons with only a pencil and ruler. Hence, with a modest understanding of geometry students can learn and solve problems in classical dynamics without calculus. An algorithm corresponding to Newton's geometrical construction is a stable analytic method for the numerical solution of equations of motion, a method that has been rediscovered several times in the past, apparently without awareness of its connection to Proposition 1.[4–8]

In Sec. II, Newton's geometrical construction for Proposition 1 is described, and is applied as a graphical method to trace orbits for central force impulses (see Fig. 1). An impulse is defined by Newton as a force acting during a very short (infinitesimal) time interval giving rise to an instantaneous change of velocity in the direction of the force. Some results are shown for impulses with various dependences on the distance from the center of force (see Fig. 2). In Sec. III, this graphical method is converted into an algorithm to integrate numerically the equations of motion of classical mechanics. In Sec. IV, an analytic proof is given of the conservation of angular momentum for discrete central force impulses, and a discrete version of energy conservation is derived. In Sec. V, it is shown that the graphical construction associated with Proposition 1 is time reversal invariant (Fig. 3). Section VI shows some examples for the rapid convergence of the graphical method (see Fig. 4). Section VII gives a proof that in the continuum limit, Newton's geometrical construction is an area preserving algorithm. In Appendix A, it is shown that the length associated with an impulse varies quadratically with the displacement length; in Appendix B, a brief historical description is given of Robert Hooke's important contribution to Newton's development of the *Principia*.

II. GRAPHICAL METHOD TO CONSTRUCT ORBITS

Newton's proof of Kepler's area law in the *Principia*, Book 1, Proposition 1, corresponding to conservation of angular momentum discussed in Corollary 1 of this proposition, is based on a geometrical construction of a discrete orbit under the action of impulses at periodic time intervals δt. By the first law of motion (inertia), between impulses the orbital velocity remains constant. The geometrical lines that represent the magnitude of the impulses end on a continuous orbital curve assumed to exist in the limits $\delta t \to 0$, the limit in which the number n of appropriately scaled impulses becomes infinite during a finite interval of time. If instead of taking these limits, the time intervals δt are kept finite, and the impulses are scaled according to the dependence on distance of the central forces, Newton's geometrical construction can be turned into a rapidly convergent graphical and numerical method to obtain approximate discrete orbits. Following Newton's description of Proposition 1, the recipe to construct graphically a polygon that describes a discrete orbit under the action of periodic impulses is the following:

In Fig. 1, panel (a) illustrates the first step in Newton's graphical construction. The horizontal line SA ends at the initial position of a body at A which is on an orbit about the center of force at S. The line AB represents the initial displacement during the first time interval δt, where $AB = v\delta t$, and v is the initial velocity. At B, AB is extended to c, with $Bc = AB$, and a line cC is drawn parallel to the radial line SB; the magnitude of cC represents the magnitude of a discrete radially inward impulse at B.[9] The line BC joining B and C is the displacement during the next time interval δt.

At the next time interval δt, this construction is repeated as shown in Fig. 1, panel (b), with the extension Cd = BC, and the impulse dD parallel to SC. This impulse is scaled in magnitude relative to cC by the radial dependence of the force law under consideration. In these diagrams, this force is assumed to be a constant. Two further constructions are shown in Fig. 1, panels (c) and (d). The latter, consisting of 4 impulses, corresponds to a very good approximation to Newton's diagram for Proposition 1. The extension of this orbit for 11 additional impulses is shown in panel (e), and in panel (f) after 72 impulses.

Given an initial displacement $d = SA$, the magnitude of the initial impulse $h = cC$ depends on the magnitude of the central force. In the diagram in Fig. 1, which evidently Newton drew carefully to scale, he chose $h \approx (1/5)d$. Subsequent values of h, i.e., Dd, Ee, Ff, are determined by the dependence of the magnitude of the impulse on the distances SC, SD, SE, SF of the vertices from the center of force at S.[10] For example, assuming a power law dependence of the impulses on distance, with exponent p, the impulse h_c at C is related to the

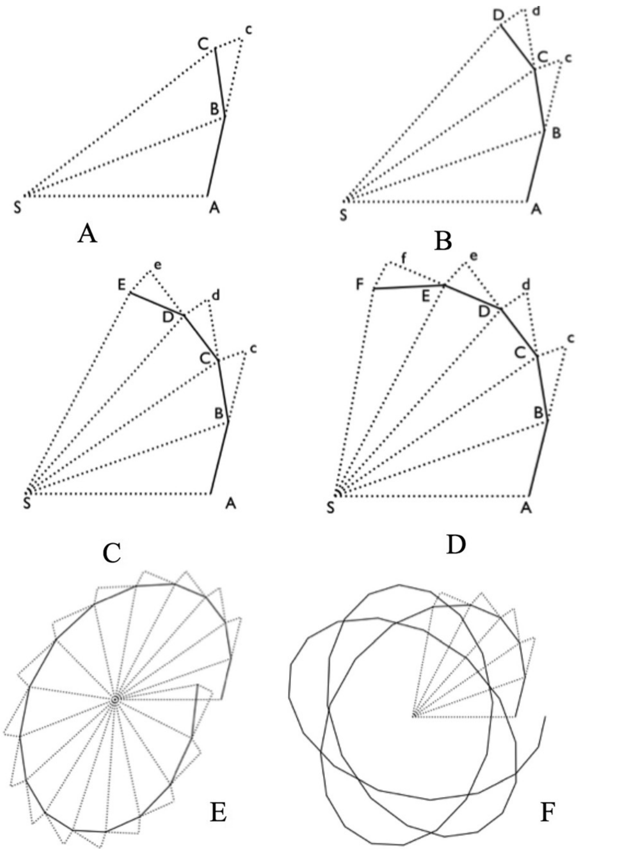

Fig. 1. Demonstration of the graphical construction based on Proposition 1. Panels (a), (b), (c), and (d) show 4 consecutive impulses in this construction that reproduces Newton's diagram for Proposition 1. Panels (e) and (f) show the orbits for a larger number of impulses.

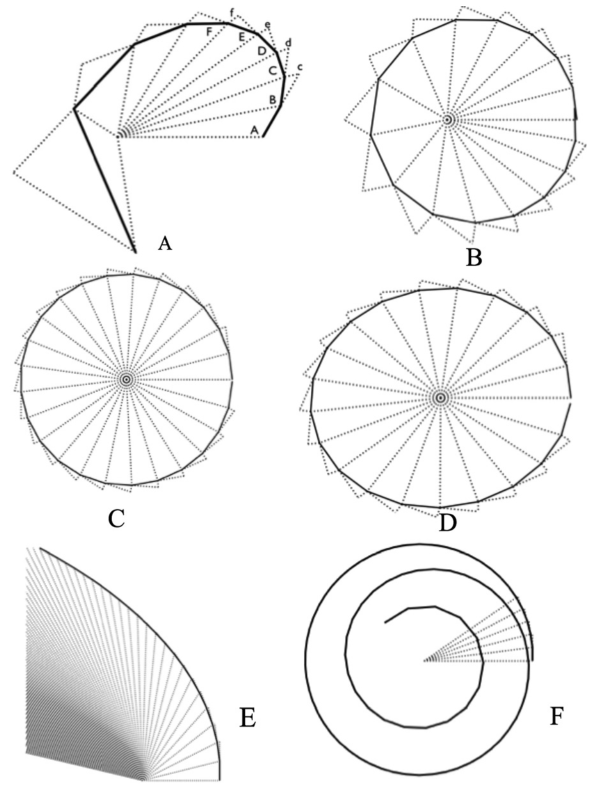

Fig. 2. Panel (a): Extension with 4 additional impacts of Newton's diagram in *De Motu* for inverse square impacts. Panel (b): Elliptical orbit for inverse square impacts. Panel (c): Circular orbit. Panel (d): Elliptical orbit for impacts dependent linearly on distance from the center. Panel (e): Hyperbolic orbit for inverse square impacts. Panel (f): Spiral orbit for inverse cubic impacts.

previous impulse h_b at B by the relation $h_c = (SC/SB)^p h_b$, and in general at any vertex F

$$h_f = (SF/SB)^p h_b. \tag{1}$$

For example, for impulses independent of distance, $p = 0$, for a linear dependence on distance, $p = 1$, and for an inverse square dependence on the distance $p = -2$.

Newton's diagram in Proposition 1 was carefully drawn to scale for *constant* impulses. This is the simplest case for a graphical computation because it avoids the need of an algebraic computation of Eq. (1) at each vertex of the discrete orbit. As Newton pointed out in a revealing letter to Edmond Halley on May 27, 1685, "I then took the simplest case for computation, which was that of Gravity uniform in a medium not Resisting."[11,12]

In Appendix A, it is shown that the magnitude of the impulse h depends quadratically on the magnitude of the displacement d. This relation is important to obtain improved discrete approximations for the orbit in the continuum limit, and follows from the local radius of curvature ρ of the orbit in this limit. For example, if the initial magnitude of d is decreased by a factor 1/2, h should be decreased by a factor 1/4, and the number of steps required to reach a comparable point of the discrete orbit is approximately doubled. Although the value of ρ is not known *a priori*, it is not required for the graphical calculation, but it can be evaluated approximately at a given vertex as the radius of a circle that contains this vertex and the two adjacent ones.[13]

Regardless of the dependence of the impulses on distance, in Proposition 1 Newton gave an elementary proof that the areas of successive triangles *SAB*, *SBC*, *SCD*, *SDE*, *SEF* obtained by joining the ends of the lines *AB*, *BC*, *CD*, *DE*, *EF* between impulses to the center at S are all equal. Since by construction these lines are path lengths transversed at equal time intervals δt, by Proposition 1 the areas enclosed by the polygonal path and the center of force are proportional to the time.[14,15]

III. CARTESIAN ALGORITHM BASED ON NEWTON'S GEOMETRICAL CONSTRUCTION

Expressing Newton's geometrical construction in Proposition 1 in algebraic form[17] leads to a very efficient computational algorithm for dynamical problems that has been rediscovered many times without the authors apparently being aware of its connection to this proposition in the *Principia*,[4–7] except for Coullet *et al.*[8]

Let $x(i)$, $y(i)$ be Cartesian coordinates for the vertices of Newton's geometrical construction at steps $i = 1, 2 \ldots n$, with the origin at the center of force. Referring to Newton's diagram in Proposition 1, Book 1 of the *Principia*, reproduced here in Fig. 1, panel (d), the initial values $x(1)$, $y(1)$ for the location of the orbiting body are $x(1) = SA$, and $y(1) = 0$. The components δx, δy of the initial displacement AB for the next position after a time interval δt determine

$$x(2) = x(1) + \delta x, \quad y(2) = y(1) + \delta y, \tag{2}$$

where $\delta x = v_x(1)\delta t$, $\delta y = v_y(1)\delta t$, and $v_x(1)$, $v_y(1)$ are the components of the initial velocity.

The next step in this algorithm is the extension of these displacements to an *auxiliary* point with coordinates $x_p(2)$, $y_p(2)$ corresponding to vertex c in Newton's diagram

$$x_p(2) = x(2) + \delta x, \quad y_p(2) = y(2) + \delta y. \tag{3}$$

For $i > 2$, these auxiliary points—d, e, f in Newton's diagram—are located at

$$x_p(i) = 2x(i) - x(i-1), \quad y_p(1) = 2y(1) - y(i-1). \tag{4}$$

The x, y components of the displacement $h(i)$, due to the impulse at step i, determine $x(i+1) = x_p(i) - h_x(i)$, $y(i+1) = y_p(i) - h_y(i)$, and substituting Eq. (4), the vertices of the resulting polygonal orbit at $x(i)$, $y(i)$ are determined successively at

$$\begin{aligned} x(i+1) &= 2x(i) - x(i-1) - h_x(i), \\ y(i+1) &= 2y(i) - y(i-1) - h_y(i), \end{aligned} \tag{5}$$

for $i = 3, 4 \ldots n$, where the auxiliary coordinates $x_p(i)$, $y_p(i)$ have now been eliminated.

With the introduction at step i of two first order differences, $\delta_- x(i) = x(i) - x(i-1)$, and $\delta_+ x(i) = x(i+1) - x(i)$, for the displacement before and after the *ith* impulse, this equation can be written in the succinct form

$$\delta_+\delta_- x(i) = -h_x(i), \quad \delta_+\delta_- y(i) = -h_y(i), \tag{6}$$

where $\delta_+\delta_- x(i)) = x(i+1) + x(i-1) - 2x(i)$, and $\delta_+\delta_- y(i)) = y(i+1) + y(i-1) - 2y(i)$. This equation shows that the displacements $h_x(i)$, $h_y(i)$ due to impulses are second order differentials.

For impulses at periodic time intervals δt,

$$\delta_+ x(i) = v_x(i+1)\delta t, \quad \delta_+ y(i) = v_y(i+1)\delta t, \tag{7}$$

where $v_x(i)$, $v_y(i)$ are the inertial velocity components *before* the impulse at step i. Setting $h_x(i) = -a_x(i)\delta t^2, h_y(i) = -a_y(i)\delta t^2$, where $a_x(i)$, $a_y(i)$ are the components of the acceleration in the continuum limit due to the impulse, we can write Eq. (6) in the form

$$\delta_+ v_x(i) = a_x(i)\delta t, \quad \delta_+ v_y(i) = a_y(i)\delta t, \tag{8}$$

for the discrete change in velocity due to the impulse at step i.

Equations (7) and (8) are the algebraic form of Newton's geometrical construction in Proposition 1, and in the continuum limit, $\delta t \to 0$, become the familiar equations of motion of classical mechanics. For finite time intervals δt, however, Eq. (7) differs in an *essential way* from the corresponding well known Euler equations for numerical integration, which have the form

$$\delta_+ x(i) = v_x(i)\delta t, \quad \delta_+ y(i) = v_y(i)\delta t, \tag{9}$$

where the velocity components $v_x(i)$, $v_y(i)$ appear at step i instead of at step $i+1$. This seemingly innocuous difference accounts for the stability and improved convergence of Newton's Proposition 1 geometrical construction both in graphical and analytic form. The construction then satisfies the conservation of angular momentum for discrete impulses, and a discrete energy conservation relation discussed in Sec. IV. Moreover, while Newton's method can be implemented graphically with the auxiliary variables $x_p(i)$, $y_p(i)$ in Eq. (4), there does not exist a corresponding graphical method to implement the Euler equations and related numerical methods of integration.

For impulses of magnitude $h(i)$ directed to the fixed center at the origin of the x, y coordinates, the components are

$$h_x(i) = h(i)\frac{x(i)}{r(i)}, \quad h_y(i) = h(i)\frac{y(i)}{r(i)}, \tag{10}$$

where $r(i) = \sqrt{(x(i)^2 + y(i)^2)}$ is the radial distance from the center of force at the *ith* step. For a power law dependence on $r(i)$, we have $h(i) = h(1)(r(i)/r(1))^p$, where $h(1)$ is the magnitude of the initial impulse, and p is the power associated with the the force law. For example, for constant impulses $p = 0$, for impulses depending linearly on the distance from the center $p = 1$, and for impulses depending inversely on the square of this distance $p = -2$.

In the special case that the impulses are directed along the initial displacement (velocity), the motion is one dimensional, and for constant impulses, $h(i) = h$, the solution of the discrete equation of motion, Eq. (6), is

$$x(i) = id + \frac{i^2 h}{2}. \tag{11}$$

By similar graphical arguments, this solution was obtained originally by Galileo, and by Isaac Beeckman, with $d = v\delta t$, and $h = a\delta t^2$, where v is the initial velocity, and the elapsed time at the *ith* step is $t = i\delta t$.[16]

IV. CONSERVATION OF ANGULAR MOMENTUM AND ENERGY IN PROPOSITION 1

The angular momentum $l(i)$ for a discrete orbit is defined by the first order difference relation

$$l(i)\delta t = x(i)\delta_- y(i) - y(i)\delta_- x(i). \tag{12}$$

Then substituting Eq. (6) for $\delta_+\delta_- x(i)$ and $\delta_+\delta_- y(i)$

$$\begin{aligned} \delta_+ l(i)\delta t = &-(x(i+1) + x(i-1))y(i) \\ &+ (y(i+1) + y(i-1))x(i) \end{aligned} \tag{13}$$

and applying Eq. (5), one obtains

$$\delta_+ l(i)\delta t = -(2x(i) - h_x(i))y(i) + (2y(i) - h_y(i))x(i). \tag{14}$$

For central forces, the impulse components $h_x(i)$, $h_y(i)$ are proportional to $x(i)$, $y(i)$, respectively, in Eq. (10) and therefore $\delta_+ l(i) = 0$.

A discrete form of energy conservation is obtained as follows: We have

$$\begin{aligned} \delta_+ v_x^2(i) &= (v_x(i+1) + v_x(i))\delta_+ v_x(i), \\ \delta_+ v_y^2(i) &= (v_y(i+1) + v_y(i))\delta v_y)i), \end{aligned} \tag{15}$$

and applying Eq. (8), we obtain

$$\delta_+ v_x^2(i) = (x(i+1) - x(i-1))a_x(i)\delta t,$$
$$\delta_+ v_y^2(i) = (v_y(i+1) + v_y(i))a_y(i)\delta t. \quad (16)$$

It follows that

$$\delta v^2(i) = 2a(i)\delta r(i), \quad (17)$$

where $v(i) = \sqrt{v_x(i)^2 + v_y(i)^2}$, $a(i) = \sqrt{a_x(i)^2 + a_y(i)^2}$, and $2\delta r(i) = (x(i)/r(i))(\delta_+ x(i) + \delta_- x(i)) + (y(i)/r(i))(\delta_+ y(i) + \delta_- y(i))$ is the mean change in the radial distance due to the *ith* impulse.

Summing both sides of Eq (17) from $i=1$ to $i=n$ gives a discrete version of the conservation of energy relation in the form

$$v(n)^2 - 2\sum_1^n a(i)\delta r(i) = v(1)^2. \quad (18)$$

In the *Principia*, Newton derived this relation in the continuum limit for one dimensional motion in Proposition 39, and applied it for two dimensional motion in Proposition 41.

V. TIME REVERSAL INVARIANCE

In Newton's geometrical construction, time reversal can be represented either by a change in sign of the periodic time intervals, $\delta t \to -\delta t$, or equivalently, by a change in sign of the velocity, $v \to -v$. In either case it will now be shown that the same vertices of the polygonal orbit are transversed but in the opposite order.

In Fig. 3, at any place along the discrete orbit with the center of force at S, let A be the position of the body at time t, and AB the displacement during the interval of time δt. At B, the impulse towards S leads to the displacement Cc parallel to the radial line SB, and the next position at C during the next interval δt is obtained by compounding (adding vectorially) Bc and Cc. Under time reversal, the velocity at C changes sign, and after a time interval δt the body reaches the previous vertex B. Following Newton's graphical construction, BC is extended to a with $Ba = BC$, and the same impulse with magnitude cC at B is now compounded (added vectorially) with Ba to a time reversed vertex A_T. Referring to Fig. 1, since Bc = AB, Ba = CB and the angles CBc and ABa at vertex B are equal, aA = Cc. Therefore $A_T = A$.

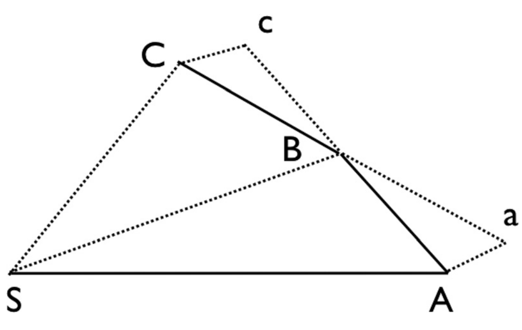

Fig. 3. Graphical demonstration that the impulse Aa under time reversal for motion from C to A is equal to the impulse Cc for motion from A to C.

VI. CONVERGENCE OF DISCRETE ORBITAL CURVES TO THE CONTINUUM LIMIT

A proof of convergence to the continuum limit of the discrete polygon obtained with Newton's geometrical construction in Proposition 1 is a difficult problem, and it is considered only partially in the *Principia*. Under the implicit assumption that the displacements representing impulses end on a *given* continuous orbital curve, Newton appealed for the existence of this limit to Lemma 3, which originally was introduced by Isaac Barrow.[18] This Lemma gives rigorous upper and lower bounds for this limit. In practice, the geometrical construction in Proposition 1 also leads to a graphical method to obtain discrete orbits that approximate the continuum curves to any desired accuracy by rescaling the magnitude of the initial parameters. Rapid convergence can be obtained by decreasing the initial displacement length d by a factor 1/2, and the initial impulse length h by a factor 1/4, and approximately doubling the number of discrete steps. Examples are shown in Fig. 4 with the initial displacements proportional to those in an early draft of the *Principia*,[20] except in Panel (b). This panel shows the convergence for inverse square impulses when the initial displacement leads to a nearly circular orbit The discrete orbits obtained with Newton's initial parameters are indicated by dash lines (red online), and the corresponding orbits obtained by rescaling these parameters by solid (blue online) lines. Panel (a) in Fig. 4 shows the convergence for a single rescaling for constant impulses, p = 0. Panel (b) shows the result for impulses depending linearly on the distance from the center of force, and panel (c) for inverse square forces. Further rescaling cannot be done graphically, because the impulse lengths become too small.

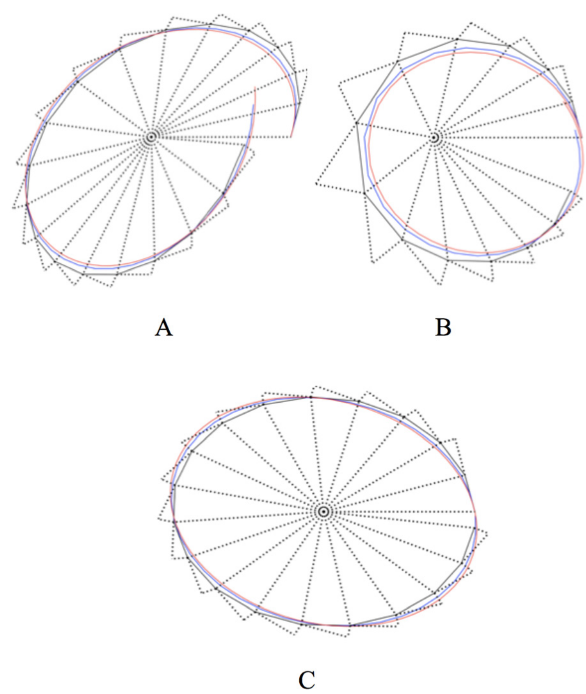

Fig. 4. Illustration of convergence. Black lines are the polygonal orbits obtained with the discrete impulse shown in the diagrams. Blue lines (color online) show orbits with initial displacement decreased by a factor 1/2, and impulses decreased by a factor 1/4. Red (color online) lines correspond to a decrease by an additional factor 1/2 and 1/4, respectively. Panel (a): Constant impulses. Panel (b): Inverse square impulses. Panel (c): Linearly dependent impulses.

VII. SYMPLECTIC ALGORITHM

In the continuum limit, Newton's graphical construction is also an area preserving algorithm. Let

$$x' = x + v\delta t = P(x, v, \delta t), \tag{19}$$

and

$$v' = v + a(x')\delta t = Q(x, v, \delta t). \tag{20}$$

After a differential time interval δt, the area $\delta x \delta v$ of a square with sides δx, δv is given by the relation

$$\left(\frac{\partial P}{\partial x}\frac{\partial Q}{\partial v} - \frac{\partial P}{\partial v}\frac{\partial Q}{\partial x}\right)\delta x \delta v. \tag{21}$$

We have

$$\frac{\partial P}{\partial x} = 1, \quad \frac{\partial Q}{\partial v} = 1 + \frac{da}{dx'}\delta t, \quad \frac{\partial P}{\partial v} = \delta t, \quad \frac{\partial Q}{\partial x} = \frac{da}{dx'}, \tag{22}$$

and hence,

$$\left(\frac{\partial P}{\partial x}\frac{\partial Q}{\partial v} - \frac{\partial P}{\partial v}\frac{\partial Q}{\partial x}\right) = 1. \tag{23}$$

VIII. FRICTION

In Book 2, Propostion 1 of the *Principia*, Newton considered the effect in a medium of resistance proportional to velocity. Such an effect can be readily taken into account in the graphical method by decreasing at each step between impulses the extension of the displacement line by some factor z. Analytically, this corresponds to replacing Eq. (4) by the relation

$$\begin{aligned} x_p(i) &= (2 - z)x(i) - (1 - z)x(i - 1), \\ y_p(i) &= (2 - z)y(1) - (1 - z)y(i - 1), \end{aligned} \tag{24}$$

where z can be determined experimentally. For example, $z \approx 0.005$ for the ball rolling in an inverted cone shown in Fig. 5, panel (b).

IX. SUMMARY

In the *Principia*, Proposition 1, Book 1, Newton gave a proof of angular momentum conservation for central forces. This proof was based on a geometrical construction that such forces consisted of periodic impulses at time intervals δt, and on taking the limit that $\delta t \to 0$, and the number of impulses $n \to \infty$. Newton justified the continuum limit by Lemma 3, Corollary 4 in the *Principia* which implied that the lines representing impulses in his geometric construction ended on a pre-existing continuous curve.[19] Perhaps for this reason the application of his construction as a graphical method to obtain such orbital curves has been overlooked. The essential feature in this application is that the magnitude of the impulse lines depends on the central force law under consideration. Expressing this graphical method in analytic form corresponds to a well known numerical method that has been rediscovered several times without awareness of its correspondence to the geometric construction in Proposition 1.

ACKNOWLEDGMENTS

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APPENDIX A: QUADRATIC DEPENDENCE OF IMPULSE ON DISPLACERMENT

Let the initial displacement $AB = d$, and the initial impuse $cC = h$. Then

$$d = v\delta t, \tag{A1}$$

where v is the initial velocity and δt is the periodic time interval. Setting δv to be the magnitude of the velocity change due to the component h_t of the impulse h *tangential* to the local motion

$$h_t = \delta v \delta t, \tag{A2}$$

δv can be expressed by a quantity a_t with the dimension of acceleration

$$\delta v = a_t \delta t. \tag{A3}$$

Substituting this expression in Eq. (A2) gives us

$$h_t = a_t \delta t^2. \tag{A4}$$

To express h_t graphically as a length, the time interval δt in this relation is replaced by d/v in Eq. (A1), giving

$$h_t = \frac{a_t d^2}{v^2}. \tag{A5}$$

If we replace v^2 by the product of a_t times a length ρ corresponding to the local radius of curvature of the continuous orbit at B,

$$\rho = \frac{v^2}{a}, \tag{A6}$$

the result is

$$h = \frac{d^2}{\rho}. \tag{A7}$$

APPENDIX B: HISTORICAL NOTE

In 1685, when Newton sent the first draft of his *Principia* entitled *De Motu Corporum Gyrum* to the Royal Society,[20] Robert Hooke who was its secretary at the time, recognized that Newton's geometrical construction could be applied as a graphical method to obtain orbits.[21,22] He proceeded to draw such an orbit for impulses that depended linearly on the distance from the center of force, and obtained a discrete polygon with the vertices located on an ellipse,[23] shown in Fig.

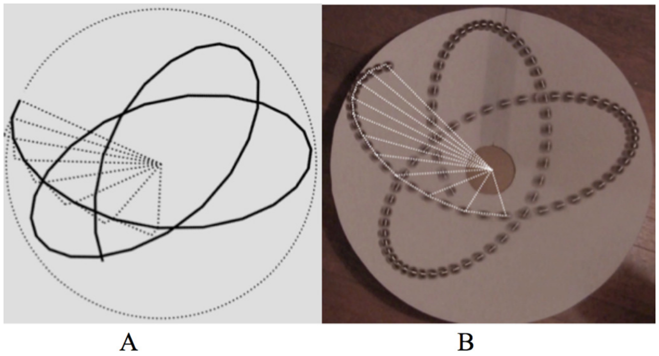


Fig. 5. Panel (a): Graphical calculation of trajectory under the action of constant impulses, and friction coefficient $z = 0.005$. Panel (b): Stroboscopic view of the trajectory of a ball rolling in an inverted cone.

6, panel (a). Six years earlier, he had communicated to Newton his own ideas about the nature of gravitational forces that accounted for planetary motion along the lines that Newton had implemented. On Dec.13, 1679 Newton had sent him a letter with the diagram shown in Fig. 6, panel (f), which presumably he obtained by a graphical method based on the local radius of curvature of an orbit[24] Hooke responded promptly: "Your calculation of the Curve by a body attracted by an equall power at all Distance from the Center Such as a ball Rouling in an inverted Concave Cone is right, and the two auges [apsides] will not unite for about a third of a Revolution."[25] In a letter to Edmond Halley on July 14, 1686, Newton admitted that, "This is true, that his Letters occasioned my finding the method of determining Figures…."[27] A reproduction of Hooke's experiment is shown in Fig. 6, panel (d).[26] In this case, the tangential force acting on the ball is a constant (apart from friction), and the graphical orbit for constant impulses shown in panel (c) is in good agreement with it. Panel (b) is the graphical orbit I obtained for inverse square dependent impulses with Hooke's initial conditions for constant impulses shown in panel (e). After seven steps the orbit diverges when the vertices approach too closely to the center of force impulses. This divergence also occurred with Newton's initial conditions in the *Principia* diagram (see Fig. 2, panel (a)). It is likely that Hooke also obtained this result, and that this divergence probably discouraged him from publishing his results.

Hooke's prompt application of Newton's geometrical construction as a graphical method to obtain an orbit gives support to a conjecture that Newton would also have applied his construction in this manner, but there isn't any evidence for it in his existing manuscripts.[28] Unfortunately Newton's work-sheets for the *Principia* have not been found. In his introduction to Newton's *Principia*, the eminent Newtonian scholar I. B. Cohen asks: "Whatever happened to the work-sheets of the *Principia*? Do they still exist in some obscure private or public collection? Was this particular set of manuscripts—alone of all the Newton papers—lost or mislaid, either when the Portsmouth Collection was still in Hurstbourne Castle or during the actual transfer to the University Library in Cambridge? Did such work-sheets still exist among Newton's papers at the time of his death? Or were they lost or destroyed—either by chance or design—during Newton's own lifetime? We may possibly never be certain of the answer to these questions."[29]

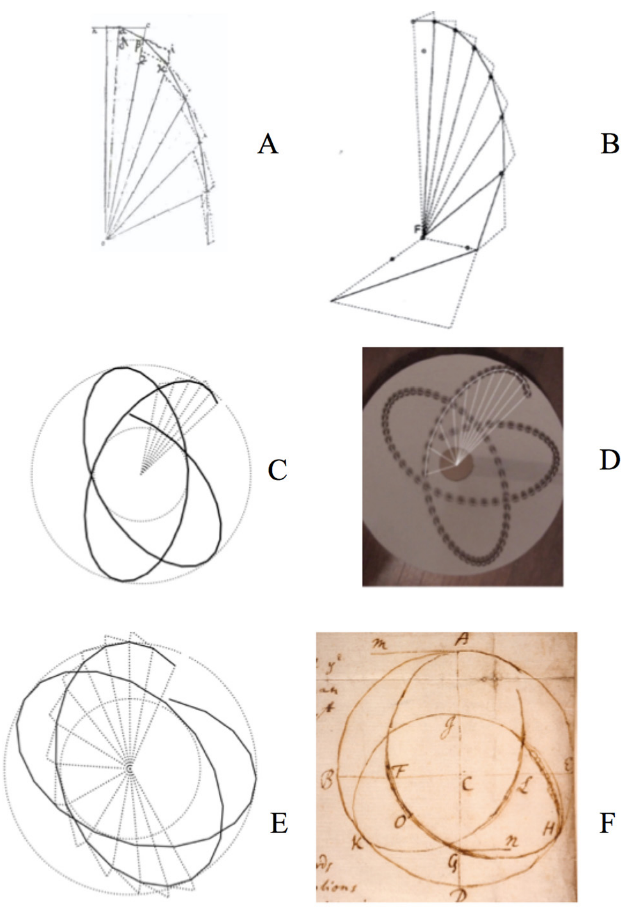


Fig. 6. Panel (a): Hooke's diagram for linear impulses in an unpublished manuscript dated September 1685. Panel (b): Diagram with Hooke's initial conditions for inverse square impulses. Panel (c): Constant impulse orbit with initial conditions approximating the experiment shown in the next panel. Panel (d): Stroboscopic view of a ball rolling in an inverted cone Ref. 25. Panel (e): Constant impulse orbit with initial conditions approximating diagram in the next panel. Panel (f): Newton's diagram in a letter to Hooke on December 13, 1679.

[1] Isaac Newton, *The Principia*, a new translation by I. B. Cohen and Anne Whitman (University of California Press, California, 1999).

[2] R. S. Westfall, *Never at Rest, A Biography of Isaac Newton* (Cambridge U.P., Cambridge, 1980), p. 470.

[3] In the *Principia* the conservation of angular momentum appears as Corollary 1 to Proposition 1, and it is applied in Proposition 41 where the angular momentum label Q is introduced without reference to Proposition 1.

[4] C. Störmer, "Méthode d'integration numérique des équations différentielles ordinaries," Compte Rendu du Congres international des mathématiciens tenu a Strasbourg du 22 au 30 September 1920–1921, pp 243–257.

[5] L. Verlet, "Computer experiments on classical fluids," Phys. Rev. **159**, 98–103 (1967).

[6] A. Cromer, "Stable solutions using the Euler approximation," Am. J. Phys. **49**, 455–459 (1981).

[7] R. P. Feynman, *The Feynman Lectures on Physics*, edited by R. P. Feynman, R. B. Leighton, and M. Sands (Addison-Wesley, Reading, Massachusetts, 1963), pp. 9-6–9-9; Feynman was familiar with Newton's proof of Kepler's area law in Proposition 1 which he presented during a lecture at Cornell in 1964 on *The Character of Physical Law* (M.I.T. Press, Cambridge, 1965), pp. 41–44.

[8] After the completion of this paper, Yves Pomeau called my attention to a publication by P. Coullet, M. Monticelli, and J. Treinert in Bulletin de l'APMEP Association des Professeurs de Mathmatiques de l'Enseignement Public (APMEP) number 450, 73–85 (2004). In their paper, these authors showed the relation between the algebraic form of Newton's Proposition 1 geometrical construction and the calculational algorithm of Stormer, Verlet, and Cromer, and they also obtained a relation for discrete energy conservation similar to that in Eq. (18).

[9]The magnitude of the displacement due to the impulse at B is cC = $a\delta t^2$, where a is the acceleration due to the central force at B, but in Proposition 1, Newton does not specify the magnitude of cC.

[10]In Proposition 1, Newton appealed to Lemma 3, Corollary 4, indicating that at each step of his geometrical construction the magnitude of the impulses are determined by a given continuous orbital curve.

[11]*The Correspondence of Isaac Newton, vol. 2, 1676–1687*, edited by H. W. Turnbull (Cambridge U.P., Cambridge, 1960), p. 433.

[12]Some renown Newtonian scholars did not realize that Newton's early methods were geometrical and based on graphical procedures. For example, Alexander Koyre remarked that "the problem (central force motion) he (Newton) deals with is very difficult, and its solution implies mathematical methods that Newton, probably did not posses at the time, perhaps not even later. Much more surprising is the very problem Newton is treating—the problem of a body submitted to a *constant* centripetal force," Isis **43**, 332 (1952).

[13]The radius of curvature of orbital curves is discussed in Proposition 6, and in Lemma 11, Principia, Book 1.

[14]In a 1964 lecture at Cornell University on the character of physical law, Richard Feynman gave a detailed presentation of Newton's geometrical proof in Proposition 1 that central forces lead to conservation of angular momentum. R. Feynman, *The Character of Physical Law* (MIT Press, Cambridge, 1965), pp. 41–43.

[15]Newton's graphical method for constant impulses is illustrated in a video at https://www.youtube.com/watch?v=5vr8p2l76ts&t=1s.

[16]The idea that accelerated motion consists of discrete changes in velocity during equal time intervals can be traced back to Galileo and to Isaac Beeckman. Galileo stated that "A motions is said to be uniformly accelerated when starting from rest, it acquires, during equal time intervals, equal increments of speed" in *Two New Sciences* (Dover Publication, New York, 1954), p. 152. According to Beeckman, the terrestrial force acting on a falling body is not truly continuous but discrete: "she pulls with small jerks." Berkel van Klass, *Isaac Beeckman on Matter and Motion, Mechanical Philosophy in the Making* (John Hopkins U.P., Maryland, 2013), p. 113.

[17]As a student at Trinity College, Cambridge University, Newton became familiar with Frans van Schooten's Latin edition of Descartes *Geometrie* that describes geometry in an algebraic form. He neglected to study Euclidian geometry which he regarded as trivial, until Isaac Barrow, during an examination, emphasized to him its importance.

[18]N. Guicciardini, *Isaac Newton on Mathematical Certainty and Method* (M.I.T. Press, Cambridge, 2000), pp. 177–1779.

[19]With the interpretation that the impulse lines in Newton's geometrical construction end on an given curve, B. Pourciau claims that Newton made "one serious (meaning fundamental and irreparable) error, assuming that his polygonal approximation argument for Proposition 1 establishes the Fixed Plane Property for centripetal motions," *The Cambridge Companion to Newton*, edited by Rob Iliffe and George Smith, 2nd ed. (Cambridge U.P., Cambridge, 2016), pp. 93–186. Actually, in a Scholium to Proposition 2 Newton remark that "if some force acts continually along a line perpendicular to the surface described it will cause the body to deviate from the plane of its motion." establishing that for central forces the orbits are confined to a plane.

[20]In 1685 Newton sent to the Royal Society an early draft of the Principia entitled De Motu Corporum Gyratum.

[21]M. Nauenberg, "Hooke, orbital motion and Newton's *Principia*," Am. J. Phys. **62**, 331–350 (1994).

[22]M. Nauenberg, "Hooke's and Newton's contributions to the early development of orbital dynamics and the theory of universal gravitation," Early Sci. Med. **10**, 518–528 (2005).

[23]Among Hooke's papers in the Trinity Library, Cambridge, there is a manuscript dated Sept. 85 that contains Hooke's drawing of an elliptical orbit.

[24]M. Nauenberg, "Newton's early computational method for dynamics," Arch. Hist. Exact Sci. bf **46**, 221–252 (1994); I. B. Cohem, "Newton's curvature measure of force," in *A Guide to Newton's Principia* (University of California Press, California, 1999), Section 3.9, pp. 78–82; "Curvature in Newton's dynamics" (with J. Brackenridge), *The Cambridge Companion to Newton*, edited by I. B. Cohen and G. Smith (Cambridge U.P., Cambridge, 2002), pp. 85–187; M. Nauenberg, "Curvature in orbital dynamics," Am. J. Phys. **73**, 340–368 (2005).

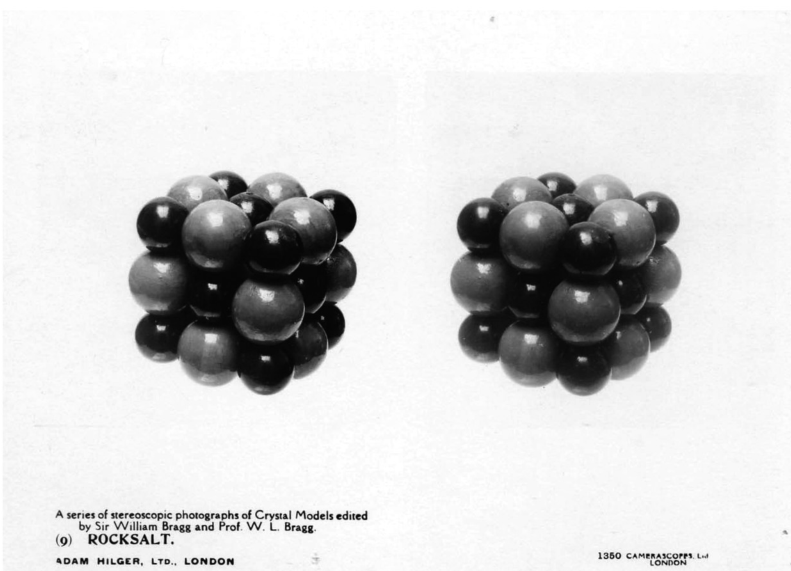
[25]Reference 20, p. 336.

[26]A video of my re-enactment of Hooke's experiment of a ball rolling in an inverted cone is shown at <https://vimeo.com/83533367>

[27]Reference 11, p. 444.

[28]All the manuscripts of Newton that have been preserved have been made available by the Cambridge Digital Library at <https://cudl.lib.cam.ac.uk/collections/newton/1>

[29]I. B. Cohen, *Introduction to Newton's Principia* (Cambridge U.P., Cambridge, 1971), p. 89.



Model of a Rock Salt Crystal

This is one of a series of thirty-five stereoscopic pictures of crystal models that were made by Sir William Bragg and his son, Prof. W.L. Bragg. This is one of two pictures of the NaCl crystal; the other one is a skeletal construction that gives the viewer a better idea of the location of the ions in the structure. The set of stereo pairs is in the Greenslade Collection. (Picture and Notes by Thomas B. Greenslade, Jr., Kenyon College)